

Stochastic comparisons of series and parallel systems with heterogeneous components

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Abstract

In this paper, we discuss stochastic comparisons of parallel systems with independent heterogeneous exponentiated Nadarajah-Haghighi (ENH) components in terms of the usual stochastic order, dispersive order, convex transform order and the likelihood ratio order. In the presence of the Archimedean copula, we study stochastic comparison of series dependent systems in terms of the usual stochastic order. Due to the great flexibility of the failure rate function of the ENH, it thus provides a good alternative to many existing life distributions in modeling positive real data sets in practice. In particular, it can be an interesting alternative to the well-known three-parameter exponentiated Weibull (EW) distribution.

Keywords: Exponentiated Nadarajah-Haghighi distribution, Stochastic ordering, Majorization, Parallel system, Series system.

1. Introduction

Let $X_{1:n} \leq \dots \leq X_{n:n}$ denote the order statistics arising from random variables X_1, \dots, X_n . Order statistics play a prominent rule in the reliability theory, life testing, operations research and other related areas. In reliability theory, the k th order statistic corresponds to the lifetime of a $(n - k + 1)$ -out-of- n system. In particular, $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively. Various researchers have studied the stochastic comparisons for the lifetimes of the series and parallel systems.

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For example [10], [6], [22], [14], [23] and [4] deal with the case of heterogeneous Weibull distributions, [7] and [11] deal with the case of heterogeneous exponentiated Weibull (EW) distributions, [1] deals with the case of heterogeneous generalized exponential (GE) distributions and [8] deals with the case of heterogeneous Frèchet distributions. A recent review on the topic can be also found in [2].

A new generalization of the exponential distribution as an alternative to the gamma, Weibull and GE distributions was proposed by Nadarajah and Haghighi [19]. Its cumulative function is given by

$$F(x) = 1 - \exp\{1 - (1 + \lambda x)^\alpha\}, \quad x > 0, \quad (1)$$

where $\lambda > 0$ is the scale parameter, and $\alpha > 0$ is the shape parameter. Lemonte [12] proposed a new three-parameter generalization of the exponential distribution on the basis of the distribution proposed by Nadarajah and Haghighi [19]. The new family of distributions is rather simple and is constructed by raising the cumulative function given in Eq. (1) to an arbitrary power, $\beta > 0$ say. The new cumulative function is given by

$$G(x) = [1 - \exp\{1 - (1 + \lambda x)^\alpha\}]^\beta, \quad x > 0, \quad (2)$$

where the parameters $\alpha > 0$ and $\beta > 0$ control the shapes of the distribution, and the parameter $\lambda > 0$ is the scale parameter. We shall refer to the new distribution given in (2) as the exponentiated NH (ENH) distribution. If a random variable X has the ENH distribution, then we write $X \sim \text{ENH}(\alpha, \lambda, \beta)$. Clearly, if $\beta = 1$, the ENH distribution reduces to the NH distribution. For $\alpha = 1$, we obtain the GE distribution proposed by Gupta and Kundu [9]. We have the exponential distribution when $\alpha = \beta = 1$.

Similarly to the exponentiated Weibull model, the ENH failure rate function can have the following four forms depending on its shape parameters: (i) increasing; (ii) decreasing; (iii) unimodal (upside-down bathtub); (iv) bathtub-shaped. Therefore, it can be used quite effectively in analyzing lifetime data. Additionally, the new ENH model can be used as an alternative to the EW distribution constructed by Mudholkar and Srivastava [18]. In Section 3, we discuss stochastic comparisons of parallel systems with independent heterogeneous ENH components in terms of the usual stochastic order, dispersive order, convex transform order and the likelihood ratio order. In the presence of the Archimedean copula, we study stochastic comparison

of series dependent systems in terms of the usual stochastic order. To continue our discussion, we need definitions of some stochastic orders and the concept of majorization which is given in Section 2 of the paper.

2. The basic definitions and some prerequisites

In this section, we recall some notions of stochastic orders, majorization and related orders and some useful lemmas, which are helpful for proving our main results. Throughout this paper, we use the notations $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_{++} = (0, +\infty)$ and the term increasing means nondecreasing and decreasing means nonincreasing.

Let X and Y be two non-negative random variables with distribution functions F and G , density functions f and g , the survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, the right continuous inverses¹ F^{-1} and G^{-1} of F and G , and hazard rate functions $h_F = \frac{f}{F}$ and $h_G = \frac{g}{G}$, respectively.

The following definition introduces some well-known orders that compare skewness of probability distributions.

Definition 2.1. X is said to be smaller than Y in the

- (i) convex transform order denoted by $X \leq_c Y$ if $G^{-1}F(x)$ is convex in $x \geq 0$;
- (ii) Lorenz order denoted by $X \leq_{\text{Lorenz}} Y$ if

$$\frac{1}{E(X)} \int_0^{F^{-1}(u)} x dx \geq \frac{1}{E(Y)} \int_0^{G^{-1}(u)} x dx, \quad \forall u \in (0, 1]. \quad (3)$$

The convex transform order implies the Lorenz order which, in turn, implies the order between the corresponding the coefficients of variations.

The following definition gives some well-known orders that compare the dispersion of two random variables.

Definition 2.2. X is said to be smaller than Y in the

- (i) dispersive order, denoted by $X \leq_{\text{disp}} Y$, if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 < \alpha \leq \beta < 1$,

¹The right continuous inverse of an increasing function h is defined as $h^{-1}(u) = \sup\{x \in \mathbb{R} : h(x) \leq u\}$

(ii) right-spread order, denoted by $X \leq_{\text{RS}} Y$, if

$$\int_{F^{-1}(u)}^{\infty} \bar{F}(x) dx \leq \int_{G^{-1}(u)}^{\infty} \bar{G}(x) dx, \quad \forall u \in (0, 1). \quad (4)$$

It is well-known that the dispersive order implies the right-spread order which, in turn, implies the order between the corresponding variances.

The next definition introduces some well-known orders that compare the magnitude of two random variables.

Definition 2.3. X is said to be smaller than Y in the

- (i) stochastic order, denoted by $X \leq_{\text{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x ;
- (ii) likelihood ratio order, denoted by $X \leq_{\text{lr}} Y$, if $g(x)/f(x)$ is increasing in $x \in \mathbb{R}_{++}$;
- (iii) hazard rate order, denoted by $X \leq_{\text{hr}} Y$, if $h_F(x) \geq h_G(x)$ for all x .

Note that the likelihood ratio order implies the hazard rate order, and the hazard rate order implies the usual stochastic order. Moreover, for non-negative random variables, the dispersive order implies the usual stochastic order. For a comprehensive discussion on various stochastic orderings, one may refer to [21] and [15].

A real function ϕ is n -monotone on $(a, b) \subseteq (-\infty, +\infty)$ if $(-1)^{n-2}\phi^{(n-2)}$ is decreasing and convex in (a, b) and $(-1)^k\phi^{(k)}(x) \geq 0$ for all $x \in (a, b)$, $k = 0, 1, \dots, n-2$, in which $\phi^{(i)}(\cdot)$ is the i th derivative of $\phi(\cdot)$. For a n -monotone ($n \geq 2$) function $\phi : [0, +\infty) \rightarrow [0, 1]$ with $\phi(0) = 1$ and $\lim_{x \rightarrow +\infty} \phi(x) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse, then

$$C_{\phi}(u_1, \dots, u_n) = \phi(\psi(u_1) + \dots + \psi(u_n)), \quad \text{for all } u_i \in [0, 1], i = 1, \dots, n,$$

is called an Archimedean copula with the generator ϕ . Archimedean copulas cover a wide range of dependence structures including the independence copula with the generator $\phi(t) = e^{-t}$. For more on Archimedean copulas, readers may refer to [20] and [17].

The concepts of majorization of vectors and Schur convexity of functions will also be needed. For some extensive and comprehensive discussions on the theory of these orders and their applications, one can see [16]. Let us recall that the notation $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ is used to denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \dots, x_n)$.

Definition 2.4. The vector \mathbf{x} is said to be

- (i) weakly submajorized by the vector \mathbf{y} (denoted by $\mathbf{x} \preceq_w \mathbf{y}$) if $\sum_{i=j}^n x_{(i)} \leq \sum_{i=j}^n y_{(i)}$ for all $j = 1, \dots, n$,
- (ii) weakly supermajorized by the vector \mathbf{y} (denoted by $\mathbf{x} \preceq^w \mathbf{y}$) if $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for all $j = 1, \dots, n$,
- (iii) majorized by the vector \mathbf{y} (denoted by $\mathbf{x} \preceq^m \mathbf{y}$) if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for all $j = 1, \dots, n-1$.

Definition 2.5. A real valued function φ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on \mathcal{A} if

$$\mathbf{x} \preceq^m \mathbf{y} \quad \text{on } \mathcal{A} \implies \varphi(\mathbf{x}) \leq (\geq) \varphi(\mathbf{y}).$$

Lemma 2.6 ([16], Theorem 3.A.8). For a function l on $\mathcal{A} \in \mathbb{R}^n$, $\mathbf{x} \preceq_w (\preceq^w) \mathbf{y}$ implies $l(\mathbf{x}) \leq l(\mathbf{y})$ if and only if it is increasing (decreasing) and Schur-convex on \mathcal{A} .

3. Main results

Firstly, we introduce the following lemma, which will be needed to prove our main results.

Lemma 3.1. Let the function $g : (1, \infty) \longrightarrow (0, \infty)$ be defined as

$$g(x) = \frac{xe^{1-x}}{1 - e^{1-x}}.$$

Then $g(x)$ is a decreasing function on $(1, \infty)$.

Proof. It is easy to see that

$$g'(x) = \frac{f(x)}{(1 - e^{1-x})^2}$$

where $f(x) = e^{1-x} - e^{2(1-x)} - xe^{1-x} = e^{1-x}(1 - x - e^{1-x}) < 0$ for $x > 1$. Hence $g(x)$ is a decreasing function on $(1, \infty)$. \square

3.1. Mutually independent samples

In this section, we provide some comparison results on the lifetimes of parallel systems arising from independent heterogeneous ENH random variables. The following result considers the comparison on the lifetimes of parallel systems in terms of the usual stochastic order with respect to the shape parameter α .

Theorem 3.2. Let X_1, \dots, X_n (X_1^*, \dots, X_n^*) be independent random variables with $X_i \sim \text{ENH}(\alpha_i, \lambda, \beta)$ ($X_i^* \sim \text{ENH}(\alpha_i^*, \lambda, \beta)$), $i = 1, \dots, n$. Then, for any $\lambda, \beta > 0$, we have

$$(\alpha_1, \dots, \alpha_n) \stackrel{w}{\succeq} (\alpha_1^*, \dots, \alpha_n^*) \implies X_{n:n} \geq_{\text{st}} X_{n:n}^*.$$

Proof. The distribution function of $X_{n:n}$ can be written as

$$G_{X_{n:n}}(x) = \prod_{i=1}^n [1 - e^{1-(1+\lambda x)^{\alpha_i}}]^{\beta}.$$

Using Lemma 2.6, it is enough to show that the function $G_{X_{n:n}}(x)$ is Schur-concave and increasing in α_i 's. The partial derivatives of $G_{X_{n:n}}(x)$ with respect to α_i is given by

$$\frac{\partial G_{X_{n:n}}(x)}{\partial \alpha_i} = \frac{\beta \log(1 + \lambda x)(1 + \lambda x)^{\alpha_i} e^{1-(1+\lambda x)^{\alpha_i}}}{1 - e^{1-(1+\lambda x)^{\alpha_i}}} G_{X_{n:n}}(x) \geq 0,$$

So, we have that $G_{X_{n:n}}(x)$ is increasing in each α_i .

To prove its Schur-concavity, it follows from Theorem 3.A.4. in [16] that we have to show that for $i \neq j$,

$$(\alpha_i - \alpha_j) \left(\frac{\partial G_{X_{n:n}}(x)}{\partial \alpha_i} - \frac{\partial G_{X_{n:n}}(x)}{\partial \alpha_j} \right) \leq 0,$$

that is, for $i \neq j$,

$$(\alpha_i - \alpha_j) G_{X_{n:n}}(x) \beta \log(1 + \lambda x) \times \left(\frac{(1 + \lambda x)^{\alpha_i} e^{1-(1+\lambda x)^{\alpha_i}}}{1 - e^{1-(1+\lambda x)^{\alpha_i}}} - \frac{(1 + \lambda x)^{\alpha_j} e^{1-(1+\lambda x)^{\alpha_j}}}{1 - e^{1-(1+\lambda x)^{\alpha_j}}} \right) \leq 0. \quad (5)$$

It is obvious that $(1 + \lambda x)^{\alpha}$ is increasing in α . So, from Lemma 3.1, we conclude that the composite function $\frac{(1+\lambda x)^{\alpha} e^{1-(1+\lambda x)^{\alpha}}}{1 - e^{1-(1+\lambda x)^{\alpha}}}$ is decreasing in α , from which it follows that (5) holds. This completes the proof of the required result. \square

We now generalize Theorem 3.2 to a wider range of the shape parameters as follows.

Theorem 3.3. Let X_1, \dots, X_n (X_1^*, \dots, X_n^*) be independent random variables with $X_i \sim \text{ENH}(\alpha_i, \lambda, \beta)$ ($X_i^* \sim \text{ENH}(\alpha_i^*, \lambda, \beta)$), $i = 1, \dots, n$. Then, for any $\lambda, \beta > 0$, if $(\alpha_1, \dots, \alpha_n) \leq (\alpha_1^*, \dots, \alpha_n^*)$, that is, $\alpha_i \leq \alpha_i^*$, $i = 1, \dots, n$, we have $X_{n:n} \geq_{\text{st}} X_{n:n}^*$.

Proof. Using the definition of the usual stochastic order and the fact that $G_{X_{n:n}}(x)$ is increasing in each α_i , the required results follow readily. \square

The following result considers the comparison on the lifetimes of parallel systems in terms of the usual stochastic order when two sets of scale parameters weakly majorize each other.

Theorem 3.4. Let X_1, \dots, X_n (X_1^*, \dots, X_n^*) be independent random variables with $X_i \sim \text{ENH}(\alpha, \lambda_i, \beta)$ ($X_i^* \sim \text{ENH}(\alpha, \lambda_i^*, \beta)$), $i = 1, \dots, n$. If $0 < \alpha \leq 1$ and $(\lambda_1, \dots, \lambda_n) \stackrel{w}{\succeq} (\lambda_1^*, \dots, \lambda_n^*)$, then $X_{n:n} \geq_{\text{st}} X_{n:n}^*$.

Proof. The distribution function of $X_{n:n}$ can be written as

$$G_{X_{n:n}}(x) = \prod_{i=1}^n [1 - e^{1-(1+\lambda_i x)^\alpha}]^\beta.$$

Using Lemma 2.6, it is enough to show that the function $G_{X_{n:n}}(x)$ is Schur-concave and increasing in λ_i 's. The partial derivatives of $G_{X_{n:n}}(x)$ with respect to λ_i is given by

$$\frac{\partial G_{X_{n:n}}(x)}{\partial \lambda_i} = \frac{x\alpha\beta(1 + \lambda_i x)^{\alpha-1} e^{1-(1+\lambda_i x)^\alpha}}{1 - e^{1-(1+\lambda_i x)^\alpha}} G_{X_{n:n}}(x) \geq 0,$$

So, $G_{X_{n:n}}(x)$ is increasing in each λ_i .

To prove its Schur-concavity, it follows from Theorem 3.A.4. in [16] that we have to show that for $i \neq j$,

$$(\lambda_i - \lambda_j) \left(\frac{\partial G_{X_{n:n}}(x)}{\partial \lambda_i} - \frac{\partial G_{X_{n:n}}(x)}{\partial \lambda_j} \right) \leq 0,$$

that is, for $i \neq j$,

$$(\lambda_i - \lambda_j) G_{X_{n:n}}(x) x \alpha \beta \left(\frac{(1 + \lambda_i x)^{\alpha-1} e^{1-(1+\lambda_i x)^\alpha}}{1 - e^{1-(1+\lambda_i x)^\alpha}} - \frac{(1 + \lambda_j x)^{\alpha-1} e^{1-(1+\lambda_j x)^\alpha}}{1 - e^{1-(1+\lambda_j x)^\alpha}} \right) \leq 0. \quad (6)$$

It is easy to show that $\frac{(1+\lambda_i x)^{\alpha-1} e^{1-(1+\lambda_i x)^\alpha}}{1-e^{1-(1+\lambda_i x)^\alpha}}$ is decreasing in λ_i for $0 < \alpha \leq 1$, from which it follows that (6) holds. This completes the proof of the result. \square

The following theorem, generalizes Theorem 3.4 to a wider range of the scale parameters.

Theorem 3.5. Let X_1, \dots, X_n (X_1^*, \dots, X_n^*) be independent random variables with $X_i \sim \text{ENH}(\alpha, \lambda_i, \beta)$ ($X_i^* \sim \text{ENH}(\alpha, \lambda_i^*, \beta)$), $i = 1, \dots, n$. If $0 < \alpha \leq 1$ and $(\lambda_1, \dots, \lambda_n) \leq (\lambda_1^*, \dots, \lambda_n^*)$, that is, $\lambda_i \leq \lambda_i^*$, $i = 1, \dots, n$, then $X_{n:n} \geq_{\text{st}} X_{n:n}^*$.

Proof. By using the definition of the usual stochastic order and the fact that $G_{X_{n:n}}(x)$ is increasing in each λ_i , the required results follow easily. \square

Now, we discuss stochastic comparison between the largest order statistics in the sense of the likelihood ratio order.

Theorem 3.6. Let X_1, \dots, X_n be independent random variables with $X_i \sim \text{ENH}(\alpha, \lambda, \beta_i)$ and X_1^*, \dots, X_n^* be another set of independent random variables with $X_i^* \sim \text{ENH}(\alpha, \lambda, \beta_i^*)$, $i = 1, \dots, n$. Then, $X_{n:n} \geq_{\text{lr}} X_{n:n}^*$ if and only if $\sum_{i=1}^n \beta_i \geq \sum_{i=1}^n \beta_i^*$.

Proof. $X_{n:n}$ has the distribution function $F_n(x) = (1 - e^{1-(1+\lambda x)^\alpha})^{\sum_{i=1}^n \beta_i}$ and $X_{n:n}^*$ has the distribution function $G_n(x) = (1 - e^{1-(1+\lambda x)^\alpha})^{\sum_{i=1}^n \beta_i^*}$, and the corresponding density functions are

$$f_n(x) = \frac{\alpha \sum_{i=1}^n \beta_i \lambda (1 + \lambda x)^{\alpha-1} e^{1-(1+\lambda x)^\alpha}}{(1 - e^{1-(1+\lambda x)^\alpha})^{1-\sum_{i=1}^n \beta_i}},$$

$$g_n(x) = \frac{\alpha \sum_{i=1}^n \beta_i^* \lambda (1 + \lambda x)^{\alpha-1} e^{1-(1+\lambda x)^\alpha}}{(1 - e^{1-(1+\lambda x)^\alpha})^{1-\sum_{i=1}^n \beta_i^*}} \quad x > 0.$$

Thus, for $x > 0$, the ratio of the density functions of $X_{n:n}$ and $X_{n:n}^*$ is $\frac{f_n(x)}{g_n(x)} = \frac{\sum_{i=1}^n \beta_i}{\sum_{i=1}^n \beta_i^*} (1 - e^{1-(1+\lambda x)^\alpha})^{\sum_{i=1}^n \beta_i - \sum_{i=1}^n \beta_i^*}$. Therefore, $\frac{f_n(x)}{g_n(x)}$ is increasing in x if and only if $\sum_{i=1}^n \beta_i \geq \sum_{i=1}^n \beta_i^*$. \square

Theorem 3.7. Let X_1, \dots, X_n be independent random variables with $X_i \sim \text{ENH}(\alpha, \lambda, \beta_i)$, and X_1^*, \dots, X_n^* be another set of independent random variables with $X_i^* \sim \text{ENH}(\alpha, \lambda, \beta_i^*)$, $i = 1, \dots, n$. Then for $\alpha < 1$ and $\lambda > 0$, we have

$$\sum_{i=1}^n \beta_i^* \leq \sum_{i=1}^n \beta_i < 1 \implies X_{n:n} \geq_{\text{disp}} X_{n:n}^*.$$

Proof. From Nadarajah and Haghighi [19], it is easy to show that $X_{n:n}$ has decreasing hazard rate (DHR) property if $\sum_{i=1}^n \beta_i < 1$ and $\alpha < 1$. Now, the desired result follows from Theorem 3.B.20 of [21] and the fact that likelihood ratio order implies hazard rate order. \square

Theorem 3.8. Suppose X_1, \dots, X_n and X_1^*, \dots, X_n^* are independent samples with $X_i \sim \text{ENH}(\alpha_1, \lambda_1, \beta_i)$, and $X_i^* \sim \text{ENH}(\alpha_2, \lambda_2, \beta_i^*)$, $i = 1, \dots, n$. Then for $\alpha_1 \leq \alpha_2$ and $\lambda > 0$, we have

$$\sum_{i=1}^n \beta_i = \sum_{i=1}^n \beta_i^* \implies X_{n:n} \geq_c X_{n:n}^*.$$

Proof. $X_{n:n}$ has the distribution function $F_{X_{n:n}}(x) = (1 - e^{1-(1+\lambda_1 x)^{\alpha_1}})^{\sum_{i=1}^n \beta_i}$ and $X_{n:n}^*$ has the distribution function $G_{X_{n:n}^*}(x) = (1 - e^{1-(1+\lambda_2 x)^{\alpha_2}})^{\sum_{i=1}^n \beta_i^*}$. Note that

$$F_{X_{n:n}}^{-1}(x) = \frac{(1 - \log(1 - x^{\frac{1}{\sum_{i=1}^n \beta_i}}))^{\frac{1}{\alpha_1}} - 1}{\lambda_1}$$

and if $\sum_{i=1}^n \beta_i = \sum_{i=1}^n \beta_i^*$ then

$$F_{X_{n:n}}^{-1}(G_{X_{n:n}^*}(x)) = \frac{(1 + \lambda_2 x)^{\frac{\alpha_2}{\alpha_1}} - 1}{\lambda_1} = h(x)$$

In order to obtain the required result it suffices to show that $h(x)$ is convex in x . The first and second partial derivatives $h(x)$ with respect to x , respectively, are

$$\frac{\partial h(x)}{\partial x} = \frac{\lambda_2}{\lambda_1} \frac{\alpha_2}{\alpha_1} (1 + \lambda_2 x)^{\frac{\alpha_2}{\alpha_1} - 1}$$

and

$$\frac{\partial^2 h(x)}{\partial x^2} = \frac{\lambda_2}{\lambda_1} \frac{\alpha_2}{\alpha_1} \lambda_2 \left(\frac{\alpha_2}{\alpha_1} - 1 \right) (1 + \lambda_2 x)^{\frac{\alpha_2}{\alpha_1} - 2}.$$

Thus, for any $\alpha_1 \leq \alpha_2$ we immediately observe that $\frac{\partial^2 h(x)}{\partial x^2}$ is positive, which completes the proof of the theorem. \square

As a direct consequence of Theorem 3.8, we present the following corollary, which is of independent interest in economics.

Corollary 3.9. Suppose X_1, \dots, X_n and X_1^*, \dots, X_n^* are independent samples with $X_i \sim \text{ENH}(\alpha_1, \lambda_1, \beta_i)$, and $X_i^* \sim \text{ENH}(\alpha_2, \lambda_2, \beta_i^*)$, $i = 1, \dots, n$. Then for $\alpha_1 \leq \alpha_2$ and $\lambda > 0$, we have

$$\sum_{i=1}^n \beta_i = \sum_{i=1}^n \beta_i^* \implies X_{n:n} \geq_{\text{Lorenz}} X_{n:n}^*.$$

3.2. Dependent samples with Archimedean structure

Recently, some efforts are made to investigate stochastic comparisons on order statistics of random variables with Archimedean copulas. See, for example, [3], [14], [13] and [5]. In this section, we derive new result on the usual stochastic order between extreme order statistics of two heterogeneous random vectors with the dependent components having exponentiated scale (ES) marginals and Archimedean copula structure. Recall that random variable X belongs to the ES family of distributions if $X \sim H(x) = [G(\lambda x)]^\alpha$, where $\alpha, \lambda > 0$ and G is called the baseline distribution and is an absolutely continuous distribution function. We denote this family by $\text{ES}(\alpha, \lambda)$. Specifically, by $\mathbf{X} \sim \text{ES}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \phi)$ we denote the sample having the Archimedean copula with generator ϕ and for $i = 1, \dots, n$, $X_i \sim \text{ES}(\alpha_i, \lambda_i)$. In the following theorem, for the ES samples with Archimedean survival copulas, we present the usual stochastic order on the smallest order statistics under weakly super-majorization order between shape parameters. For $\mathbf{X} \sim \text{ES}(\boldsymbol{\alpha}, \lambda, \phi)$ and $\mathbf{X}^* \sim \text{ES}(\boldsymbol{\alpha}^*, \lambda, \phi)$, Bashkar et al. [3] in Theorem 4.1 showed that $X_{1:n} \leq_{\text{st}} X_{1:n}^*$ if $\boldsymbol{\alpha} \stackrel{w}{\succeq} \boldsymbol{\alpha}^*$. Theorem 3.10 generalizes the result of [3] to ES samples with not necessarily a common dependence structure. The smallest order statistic $X_{1:n}$ of the sample $\mathbf{X} \sim \text{ES}(\boldsymbol{\alpha}, \lambda, \phi)$ gets survival function

$$\bar{G}_{X_{1:n}}(x) = \phi\left(\sum_{i=1}^n \psi(1 - G^{\alpha_i}(\lambda x))\right) = J(\boldsymbol{\alpha}, \lambda, x, \phi) \quad (7)$$

Theorem 3.10. For $\mathbf{X} \sim \text{ES}(\boldsymbol{\alpha}, \lambda, \phi_1)$ and $\mathbf{X}^* \sim \text{ES}(\boldsymbol{\alpha}^*, \lambda, \phi_2)$, if $\psi_2 \circ \phi_1$ is super-additive, then $\boldsymbol{\alpha} \stackrel{w}{\succeq} \boldsymbol{\alpha}^*$ implies $X_{1:n} \leq_{\text{st}} X_{1:n}^*$.

Proof. According to Equation (7), $X_{1:n}$ and $X_{1:n}^*$ have their respective survival functions $J(\boldsymbol{\alpha}, \lambda, x, \phi_1)$ and $J(\boldsymbol{\alpha}^*, \lambda, x, \phi_2)$ for $x \geq 0$.

First we show that $J(\boldsymbol{\alpha}, \lambda, x, \phi_1)$ is increasing and Schur-concave function of $\alpha_i, i = 1, \dots, n$. Since ϕ_1 is decreasing, we have

$$\frac{\partial J(\boldsymbol{\alpha}, \lambda, x, \phi_1)}{\partial \alpha_i} = - \frac{F^{\alpha_i}(\lambda x) \log(F(\lambda x)) \phi_1'(\sum_{i=1}^n \psi(1 - F^{\alpha_i}(\lambda x)))}{\phi_1'(\psi(1 - F^{\alpha_i}(\lambda x)))} \geq 0,$$

for all $x > 0$,

That is, $J(\boldsymbol{\alpha}, \lambda, x, \phi_1)$ is increasing in α_i for $i = 1, \dots, n$.

To prove its Schur-concavity, it follows from Theorem 3.A.4. in [16] that we have to show that for $i \neq j$,

$$(\alpha_i - \alpha_j) \left(\frac{\partial J(\boldsymbol{\alpha}, \lambda, x, \phi_1)}{\partial \alpha_i} - \frac{\partial J(\boldsymbol{\alpha}, \lambda, x, \phi_1)}{\partial \alpha_j} \right) \leq 0,$$

that is, for $i \neq j$,

$$\begin{aligned} & -\log(F(\lambda x)) \phi_1' \left(\sum_{i=1}^n \psi_1(1 - F^{\alpha_i}(\lambda x)) \right) (\alpha_i - \alpha_j) \\ & \left(\frac{F^{\alpha_i}(\lambda x)}{\phi_1'(\psi_1(1 - F^{\alpha_i}(\lambda x)))} - \frac{F^{\alpha_j}(\lambda x)}{\phi_1'(\psi_1(1 - F^{\alpha_j}(\lambda x)))} \right) \leq 0. \end{aligned} \quad (8)$$

Now, let us consider the function $g(\alpha) = \frac{F^\alpha(\lambda x)}{\phi'(\psi(1 - F^\alpha(\lambda x)))}$. Taking derivative with respect to α , we get

$$\begin{aligned} g'(\alpha) & \stackrel{\text{sgn}}{=} F^\alpha(\lambda x) \log(F(\lambda x)) \phi'(\psi(1 - F^\alpha(\lambda x))) \\ & + \frac{F^{2\alpha}(\lambda x) \log(F(\lambda x))}{\phi'(\psi(1 - F^\alpha(\lambda x)))} \phi''(\psi(1 - F^\alpha(\lambda x))) \geq 0. \end{aligned}$$

Thus, $g(\alpha)$ is increasing with respect to α , from which it follows that (8) holds. According to Lemma 2.6 $\boldsymbol{\alpha} \stackrel{w}{\succeq} \boldsymbol{\alpha}^*$ implies $J(\boldsymbol{\alpha}, \lambda, x, \phi_1) \leq J(\boldsymbol{\alpha}^*, \lambda, x, \phi_1)$. On the other hand, since $\psi_2 \circ \phi_1$ is super-additive by Lemma A.1. of [13], we have $J(\boldsymbol{\alpha}^*, \lambda, x, \phi_1) \leq J(\boldsymbol{\alpha}^*, \lambda, x, \phi_2)$. So, it holds that

$$J(\boldsymbol{\alpha}, \lambda, x, \phi_1) \leq J(\boldsymbol{\alpha}^*, \lambda, x, \phi_1) \leq J(\boldsymbol{\alpha}^*, \lambda, x, \phi_2).$$

That is, $X_{1:n} \leq_{\text{st}} X_{1:n}^*$. □

Note that if in Theorem 3.10, we take $\lambda = 1$, then we get the following result for the proportional reversed hazards (PRH) model.

Corollary 3.11. Suppose $\mathbf{X} \sim PRH(\boldsymbol{\alpha}, \phi_1)$ and $\mathbf{X}^* \sim PRH(\boldsymbol{\alpha}^*, \phi_2)$ and $\phi_2 \circ \psi_1$ is super-additive. Then $\boldsymbol{\alpha} \stackrel{w}{\succeq} \boldsymbol{\alpha}^*$ implies $X_{1:n} \leq_{\text{st}} X_{1:n}^*$.

The following corollary immediately follows from the above theorem.

Corollary 3.12. Suppose $\mathbf{X} \sim ENH(\alpha, \lambda, \boldsymbol{\beta}, \phi_1)$ and $\mathbf{X}^* \sim ENH(\alpha, \lambda, \boldsymbol{\beta}^*, \phi_2)$ and $\phi_2 \circ \psi_1$ is super-additive. Then $\boldsymbol{\beta} \stackrel{w}{\succeq} \boldsymbol{\beta}^*$ implies $X_{1:n} \leq_{\text{st}} X_{1:n}^*$.

4. Conclusions

The failure rate function of the ENH model can be constant, decreasing, increasing, upside-down bathtub (unimodal) and bathtub-shaped. Due to the great flexibility of the failure rate function of this model, it thus provides a good alternative to many existing life distributions in modeling positive real data sets in practice. In particular, it can be an interesting alternative to the well-known three-parameter EW distribution. In this paper, we discussed stochastic comparisons of parallel systems with independent heterogeneous ENH components in terms of the usual stochastic order, dispersive order, convex transform order and the likelihood ratio order. In the presence of the Archimedean copula, we studied stochastic comparison of series dependent systems in terms of the usual stochastic order.

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